

# AN UPPER BOUND FOR THE SIZE OF A $k$ -UNIFORM INTERSECTING FAMILY WITH COVERING NUMBER $k$ .

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ABSTRACT. Let  $r(k)$  denote the maximum number of edges in a  $k$ -uniform intersecting family with covering number  $k$ . Erdős and Lovász proved that  $\lfloor k!(e-1) \rfloor \leq r(k) \leq k^k$ . Frankl, Ota, and Tokushige improved the lower bound to  $r(k) \geq (k/2)^{k-1}$ , and Tuza improved the upper bound to  $r(k) \leq (1 - e^{-1} + o(1))k^k$ . We establish that  $r(k) \leq (1 + o(1))k^{k-1}$ .

## 1. INTRODUCTION

Let  $X$  be a finite set and  $k$  be a positive integer. A family of sets  $\mathcal{F} \subseteq \binom{X}{k}$  is called a  $k$ -uniform hypergraph, or a  $k$ -uniform family. The hypergraph  $\mathcal{F}$  is intersecting if all  $e_1, e_2 \in \mathcal{F}$  satisfy  $e_1 \cap e_2 \neq \emptyset$ . A set  $C \subseteq X$  is called a cover of  $\mathcal{F}$  if every  $f \in \mathcal{F}$  satisfies  $f \cap C \neq \emptyset$ . The covering number of  $\mathcal{F}$ , denoted by  $\tau(\mathcal{F})$ , is define by  $\tau(\mathcal{F}) := \min\{|C| : C \text{ is a cover of } \mathcal{F}\}$ .

Define

$$r(k) := \max\{|\mathcal{F}| : \mathcal{F} \text{ is } k\text{-uniform, intersecting, and } \tau(\mathcal{F}) = k\},$$

where no restriction is placed upon the size of the vertex set  $X$ .

In 1975, Erdős and Lovász [2] proved that

$$\lfloor k!(e-1) \rfloor \leq r(k) \leq k^k.$$

In 1994, Tuza [3] improved the upper bound, and in 1996, Frankl, Ota, and Tokushige [1] improved the lower bound. It follows from these result that

$$\left(\frac{k}{2}\right)^{k-1} \leq r(k) \leq (1 - e^{-1} + o(1))k^k.$$

Our main result is an improved upper bound. This will be established by using the following two lemmas, which will be proved in Sections 3 and 4, respectively. The first lemma is based upon the degree of a vertex  $x \in X$ , denoted  $d(x)$ , which is the number of elements in  $\mathcal{F}$  that contain  $x$ .

**Lemma 1.** *Let  $\mathcal{F}$  be a  $k$ -uniform intersecting family on  $X$  with covering number  $k$ . If  $x \in X$  satisfies  $d(x) \geq (\log k)k^{k-2}$ , then*

$$|\{f \in \mathcal{F} : f \not\ni x\}| = o(k^{k-1}).$$

The next lemma is based upon the maximum degree of a hypergraph  $\mathcal{F}$  on  $X$ , which is defined by  $\Delta(\mathcal{F}) := \max\{d(x) : x \in X\}$ .

**Lemma 2.** *Let  $\mathcal{F}$  be a  $k$ -uniform intersecting family on  $X$  with covering number  $k$ . Let  $\alpha \in \mathbb{R}^+$ . If  $\Delta(\mathcal{F}) \leq |\mathcal{F}|/40\alpha \log k$ , then for  $k$  sufficiently large*

$$|\mathcal{F}| \leq \max\{2k^{2k/3}, ek^{k-\alpha}\}.$$

Together, these two lemmas will be used to prove our main result.

**Theorem 3.** *The function  $r(k)$  satisfies*

$$r(k) \leq (1 + o(1))k^{k-1}.$$

*Proof.* Let  $\mathcal{F}$  be a  $k$ -uniform intersecting family on  $X$  with covering number  $k$ . We consider two cases.

If  $\Delta(\mathcal{F}) \geq (\log k)k^{k-2}$ , let  $x \in X$  be a vertex with  $d(x) \geq (\log k)k^{k-2}$ . A simple observation (which follows from Lemma 5), is that any  $k$ -uniform intersecting family  $\mathcal{F}$  with covering number  $k$  satisfies  $\Delta(\mathcal{F}) \leq k^{k-1}$ . From this observation and Lemma 1,

$$|\mathcal{F}| \leq d(x) + |\{f \in \mathcal{F} : f \not\ni x\}| \leq k^{k-1} + o(k^{k-1}),$$

as desired.

In the complementary case  $\Delta(\mathcal{F}) < (\log k)k^{k-2}$ , we proceed by contradiction. That is, assume that  $\Delta(\mathcal{F}) < (\log k)k^{k-2}$  and that  $|\mathcal{F}| > k^{k-1}$ . For  $\alpha = k/40 \log^2 k$ , we have that

$$\Delta(\mathcal{F}) < (\log k)k^{k-2} \leq \frac{|\mathcal{F}|}{40\alpha \log k},$$

and hence Lemma 2 gives that for  $k$  sufficiently large

$$|\mathcal{F}| \leq \max\{2k^{2k/3}, ek^{k-\alpha}\} < k^{k-1},$$

completing the proof. □

Lemmas 1 and 2 will be established in Sections 3 and 4, respectively. The next section will introduce some notation, a pair of general lemmas, and a Guesser-Chooser game upon which the proofs of Lemmas 1 and 2 will be based.

## 2. PRELIMINARIES

We will use the following notation. For  $\tilde{\mathcal{F}} \subseteq \mathcal{F}$  and  $S \subset X$ , the degree of  $S$  in  $\tilde{\mathcal{F}}$ , denoted by  $d_{\tilde{\mathcal{F}}}(S)$ , is defined by  $d_{\tilde{\mathcal{F}}}(S) := |\{f \in \tilde{\mathcal{F}} : f \supseteq S\}|$ . We also take  $d(S) := d_{\mathcal{F}}(S)$ . For integers  $i$  and  $j$ , let  $[i] := \{1, 2, \dots, i\}$ , let  $[i, j] := [j] \setminus [i-1]$ , and let  $(i, j] := [j] \setminus [i]$ . We write  $(\log k - i)$  to stand for  $(\log(k) - i)$ .

We begin by establishing the following lemma.

**Lemma 4.** *Let  $\mathcal{F}$  be a  $k$ -uniform intersecting family on  $X$  with covering number  $k$  and let  $\tilde{\mathcal{F}} \subseteq \mathcal{F}$ . Let  $j \in [k]$  and let  $S_{j-1}$  be any subset of  $X$  with size  $|S_{j-1}| = j-1$ . Then there exists  $S_j = \{s_j\} \cup S_{j-1}$  with  $|S_j| = j$  such that*

$$d_{\tilde{\mathcal{F}}}(S_j) \geq k^{-1} \cdot d_{\tilde{\mathcal{F}}}(S_{j-1}).$$

*Proof.* Since  $\mathcal{F}$  has covering number  $k$  and  $|S_{j-1}| < k$ , there is an edge  $f \in \mathcal{F}$  such that  $f \cap S_{j-1} = \emptyset$ . Because  $\mathcal{F}$  is an intersecting family,

$$\sum_{x \in f} d_{\tilde{\mathcal{F}}}(S_{j-1} \cup x) \geq d_{\tilde{\mathcal{F}}}(S_{j-1}).$$

Therefore, for some  $\tilde{x} \in f$ , we have  $d_{\tilde{\mathcal{F}}}(S_{j-1} \cup \tilde{x}) \geq k^{-1} \cdot d_{\tilde{\mathcal{F}}}(S_{j-1})$ . It suffices to take  $s_j := \tilde{x}$ .  $\square$

We will also make use of the following lemma.

**Lemma 5.** *Let  $\mathcal{F}$  be a  $k$ -uniform intersecting family on  $X$  with covering number  $k$ . If  $U \subset X$  with  $|U| = u$ , then  $d(U) \leq k^{k-u}$ .*

*Proof.* We induct on  $u$ . If  $u = k$ , then  $d(U) \leq 1$ . For  $u < k$ , choose  $f \in \mathcal{F}$  such that  $f \cap U = \emptyset$ ; such an edge exists since  $\tau(\mathcal{F}) = k > u$ . Making use of the fact that every edge containing  $U$  must intersect  $f$  and our inductive hypothesis,

$$d(U) \leq \sum_{x \in f} d(U \cup \{x\}) \leq k \cdot k^{k-(u+1)} = k^{k-u},$$

completing the proof.  $\square$

For  $U = \emptyset$ , this yields

$$r(k) \leq k^k, \tag{1}$$

as first proved by Erdős and Lovász in [2]. We now give another proof of (1) in order to introduce some of the key ideas and notation that will be used in the proofs of Lemmas 1 and 2.

*Guesser-Chooser proof of equation (1).* We consider a game played between a Chooser and a Guesser. The game is played on a fixed hypergraph  $\mathcal{F}$ , which is known to both players. The Chooser selects an edge  $e \in \mathcal{F}$  which is not revealed to Guesser. Guesser then asks a sequence of questions  $\Omega_1, \Omega_2, \dots, \Omega_k$  to gain information about the edge  $e$ . Each question  $\Omega_i$  must have a unique answer  $\omega_i \in [k]$ . If Guesser can always determine the edge  $e$  after asking  $k$  such questions, it follows that  $|\mathcal{F}| \leq k^k$ . Equivalently, this can be thought of as creating an injection from  $\mathcal{F}$  to the set of all sequences of the form  $\omega_1, \omega_2, \dots, \omega_k$  where  $\omega_i \in [k]$  for all  $i \in [k]$ .

We remark that in this game, the questions Guesser asks may depend on the answers to the previous questions, but can not depend on knowledge of the edge  $e$  that is not available to Guesser.

We now describe such a  $k$  question strategy for Guesser. Guesser first selects an arbitrary edge  $e_1 \in \mathcal{F}$  and fixes an arbitrary labeling  $e_1 = \{e_1^1, e_2^1, \dots, e_k^1\}$ . Question  $\Omega_1$  asks for the least  $\omega_1$  such that  $e_{\omega_1}^1 \in e$ ; indeed, since  $\mathcal{F}$  is a  $k$ -uniform intersecting family, there is a unique answer  $\omega_1 \in [k]$ . Hence, our first question identifies one vertex  $e_{\omega_1}^1 \in e$ .

More generally, question  $\Omega_i$  is determined as follows. Guesser selects an edge  $e_i \in \mathcal{F}$  that does not intersect  $\{e_{\omega_1}^1, e_{\omega_2}^2, \dots, e_{\omega_{i-1}}^{i-1}\}$ , which exists since  $\tau(\mathcal{F}) = k$ . Guesser then fixes an arbitrary labeling  $e_i = \{e_1^i, e_2^i, \dots, e_k^i\}$ . Question  $\Omega_i$  asks for the least  $\omega_i$  such that  $e_{\omega_i}^i \in e$ . Hence, after  $k$  questions are asked, Guesser has determined  $e = \{e_{\omega_1}^1, e_{\omega_2}^2, \dots, e_{\omega_k}^k\}$ .  $\square$

### 3. PROOF OF LEMMA 1

Let  $\mathcal{F}$  be a  $k$ -uniform intersecting family on  $X$  with  $\tau(\mathcal{F}) = k$ . Let  $x \in X$  with  $d(x) \geq (\log k)k^{k-2}$ . Let

$$t := \lfloor \log k \rfloor.$$

To show  $|\{f \in \mathcal{F} : f \not\ni x\}| \leq k^{k-1}$ , we will make use of the Guesser-Chooser game introduced in the Guesser-Chooser proof of Equation (1). Chooser will select an edge  $e \in \mathcal{F}$  with  $e \not\ni x$  and then Guesser will ask a sequence of  $k$  questions  $\Omega_1, \Omega_2, \dots, \Omega_k$  that will yield corresponding answers  $\omega_1, \omega_2, \dots, \omega_k$  with  $\omega_i \in [k]$  for all  $i \in [k]$ . Unlike the previous proof, Guesser will now choose his questions so that the first  $t$  answers form a non-decreasing sequence, that is

$$\omega_1 \leq \omega_2 \leq \dots \leq \omega_t.$$

The key idea to our proof is that for  $i \in [t]$ , after having asked questions  $\Omega_1, \Omega_2, \dots, \Omega_i$  and received answers  $\omega_1, \omega_2, \dots, \omega_i$ , Guesser will have determined

- a set  $V_i \subset e$  with  $|V_i| = i$ ,
- a set  $U_i \subset X \setminus e$  with  $|U_i| = \omega_i - 1$ , and
- a collection of edges  $\mathcal{F}_i := \{f \in \mathcal{F} : f \supseteq U_i \text{ and } f \cap V_i = \emptyset\}$  with

$$|\mathcal{F}_i| \geq (\log k - i)k^{k-\omega_i}. \quad (2)$$

We will say that the sets  $V_i$  and  $U_i$  exhibit property  $\mathcal{P}_i$  if all three of the above criteria are satisfied. Let  $V_0 := \emptyset$ , let  $U_0 := \{x\}$ , and let  $\omega_0 := 2$ . It follows that  $|\mathcal{F}_0| = d(x) \geq (\log k)k^{k-2}$ . Observe that Guesser knows that  $V_0$  and  $U_0$  exhibit property  $\mathcal{P}_0$ .

**Claim 6.** *Let  $i \in [t]$ . Given sets  $V_{i-1}$  and  $U_{i-1}$  exhibiting  $\mathcal{P}_{i-1}$ , Guesser can ask a question  $\Omega_i$  whose answer  $\omega_i$  will determine sets  $V_i$  and  $U_i$  exhibiting property  $\mathcal{P}_i$ . Moreover, Guesser can guarantee that  $\omega_i \geq \omega_{i-1}$ .*

*Proof.* We will specify an edge  $e_i = \{e_1^i, e_2^i, \dots, e_k^i\}$ . Question  $\Omega_i$  will then ask for the least  $\omega_i$  such that  $e_{\omega_i}^i \in e$ .

Fix a labeling  $U_{i-1} = \{u_1, \dots, u_{\omega_{i-1}-1}\}$ . For  $j \in [\omega_{i-1} - 1]$ , take  $e_j^i := u_j$ . This will ensure that  $\omega_i \geq \omega_{i-1}$  as desired, since  $U_{i-1} \cap e = \emptyset$ .

Let  $S_{\omega_{i-1}}^i := U_{i-1}$ . We now proceed recursively as follows: for  $j \in [\omega_{i-1}, k]$ , apply Lemma 4 to  $S_{j-1}^i$  with respect to  $\mathcal{F}_{i-1}$  to obtain  $S_j^i = S_{j-1}^i \cup \{e_j^i\}$ . For  $j \in [\omega_{i-1}, k]$ , this yields sets  $S_j^i$  with

$$d_{\mathcal{F}_{i-1}}(S_j^i) \geq k^{-j+\omega_{i-1}-1} d_{\mathcal{F}_{i-1}}(S_{\omega_{i-1}}^i) = k^{-j+\omega_{i-1}-1} |\mathcal{F}_{i-1}|. \quad (3)$$

By (2),

$$k^{-j+\omega_{i-1}-1} |\mathcal{F}_{i-1}| \geq (\log k - i + 1) k^{k-j-1}. \quad (4)$$

It follows from (3) and (4) that for  $j \in [\omega_{i-1}, k]$ ,

$$d_{\mathcal{F}_{i-1}}(S_j^i) \geq (\log k - i + 1) k^{k-j-1}. \quad (5)$$

Now making use of  $i \leq t$ , from (5) we have that  $d_{\mathcal{F}_{i-1}}(S_k^i) > 0$ . From the definition of  $\mathcal{F}_{i-1}$ , it now follows that  $S_k^i \cap V_{i-1} = \emptyset$ . Hence,  $e_j^i \notin V_{i-1}$  for all  $j \in [k]$ .

Having completed our construction of  $e_i$ , we now consider the answer  $\omega_i$  to question  $\Omega_i$ . Define

$$V_i := V_{i-1} \cup \{e_{\omega_i}^i\} \quad \text{and} \quad U_i := \{e_1^i, e_2^i, \dots, e_{\omega_i-1}^i\}.$$

Observe that  $\mathcal{F}_i$  is precisely the edges in  $\mathcal{F}_{i-1}$  that contain  $U_i = S_{\omega_{i-1}}^i$  and do not contain  $e_{\omega_i}^i$ . Making use of (5) and Lemma 5,

$$\begin{aligned} |\mathcal{F}_i| &\geq d_{\mathcal{F}_{i-1}}(S_{\omega_{i-1}}^i) - d_{\mathcal{F}}(S_{\omega_i}^i \cup \{e_{\omega_i}^i\}) \\ &\geq (\log k - i + 1)k^{k-\omega_i} - k^{k-\omega_i} \\ &= (\log k - i)k^{k-\omega_i}. \end{aligned}$$

Thus, we have shown that  $V_i$  and  $U_i$  exhibit property  $\mathcal{P}_i$ .  $\square$

It follows from Claim 6 that Guesser may ask questions,  $\Omega_1, \Omega_2, \dots, \Omega_t$  that necessitate a non-decreasing sequence of answers  $\omega_1, \omega_2, \dots, \omega_t$ . Moreover, after asking these questions, Guesser will have determined  $V_t \subset e$  with  $|V_t| = t$ . For the remaining  $k - t$  questions, Guesser will no longer ask questions that necessitate a non-decreasing sequence.

**Claim 7.** *Let  $i \in (t, k]$ . Given a set  $V_{i-1} \subseteq e$  with  $|V_{i-1}| = i - 1$ , Guesser can ask a question  $\Omega_i$  whose answer  $\omega_i$  will allow Guesser to determine a set  $V_i \subseteq e$  with  $|V_i| = i$ .*

*Proof.* Let  $e_i$  be any edge not covered by  $V_{i-1}$ ; such an edge exists since  $\tau(\mathcal{F}) > i - 1$ . Arbitrarily label  $e_i = \{e_1^i, e_2^i, \dots, e_k^i\}$ . Question  $\Omega_i$  asks for the least  $\omega_i$  such that  $e_{\omega_i}^i \in e$ . Let  $V_i := V_{i-1} \cup \{e_{\omega_i}^i\}$ .  $\square$

Hence, after  $k$  questions are asked, Guesser will have determined  $e = V_k$ . Since the first  $t$  answers are non-decreasing and the number of non-decreasing sequences in  $[k]^t$  is  $\binom{k+t-1}{t}$ , this gives that

$$\begin{aligned} |\{f \in \mathcal{F} : f \not\supseteq x\}| &\leq \binom{k+t-1}{t} k^{k-t} \\ &\leq \left(\frac{e(k+t-1)}{t}\right)^t k^{k-t} \\ &\leq \left(\frac{e}{t} \left(1 + \frac{t-1}{k}\right)\right)^t k^k \\ &\leq \left(\frac{2e}{t}\right)^t k^k \leq \left(\frac{2e}{\log k - 1}\right)^{\log k - 1} k^k \leq k^{k - (1 - o(1)) \log \log k} = o(k^{k-1}). \end{aligned}$$

This completes the proof of Lemma 1.

## 4. PROOF OF LEMMA 2

Let  $\mathcal{F}$  be a  $k$ -uniform intersecting family on  $X$  with  $\tau(\mathcal{F}) = k$ . Suppose that  $\Delta(\mathcal{F}) \leq |\mathcal{F}|/40\alpha \log k$ . To prove Lemma 2, it suffices to prove that if  $|\mathcal{F}| > 2k^{2k/3}$ , then  $|\mathcal{F}| \leq ek^{k-\alpha}$ . Hence, we assume that  $|\mathcal{F}| > 2k^{2k/3}$ .

Let

$$t := 20\lfloor \alpha \log k \rfloor. \quad (6)$$

As in the proofs of Equation (1) and Theorem 1, we will make use of the Guesser and Chooser game. As before, Chooser will select an edge  $e \in \mathcal{F}$  and then Guesser will ask a sequence of  $k$  questions  $\Omega_1, \Omega_2, \dots, \Omega_k$  that will yield corresponding answers  $\omega_1, \omega_2, \dots, \omega_k$  with  $\omega_i \in [k]$  for all  $i \in [k]$ . Unlike the previous two proofs, Guesser will now choose his questions so that

$$\omega_i > 2k/3 \implies \omega_{i+1} > k/3 \quad \text{for all odd } i \in [t]. \quad (7)$$

Let  $V_0 := \emptyset$ . The following claim establishes that Guesser can ask his first  $t$  questions so that (7) is satisfied.

**Claim 8.** *Let  $i \in [t]$  be an odd number. Given a set  $V_{i-1} \subset e$  with  $|V_{i-1}| = i - 1$ , Guesser can ask a pair of questions  $\Omega_i$  and  $\Omega_{i+1}$  whose answers will determine a set  $V_{i+1} \subset e$  with  $|V_{i+1}| = i + 1$ . Moreover, these questions can be asked so that  $\omega_i > 2k/3$  implies that  $\omega_{i+1} > k/3$ .*

*Proof.* Let  $i \in [t]$  be an odd number and  $V_{i-1} \subset e$  with  $|V_{i-1}| = i - 1$ . Let

$$\mathcal{F}_{i-1} := \{f \in \mathcal{F} : f \cap V_{i-1} = \emptyset\}.$$

It follows that

$$|\mathcal{F}_{i-1}| \geq |\mathcal{F}| - \Delta(\mathcal{F}) \cdot (i - 1) \geq |\mathcal{F}|/2 \geq k^{2k/3}.$$

We now construct a testing edge  $e_i = \{e_1^i, e_2^i, \dots, e_k^i\}$ . We begin by specifying the first  $\lfloor k/3 \rfloor$  vertices in  $e_i$  as follows. Let  $S_0^i := \emptyset$ . We now proceed recursively: for  $j \in [\lfloor k/3 \rfloor]$ , apply Lemma 4 to  $S_{j-1}^i$  with respect to  $\mathcal{F}_{i-1}$  to obtain  $S_j^i = S_{j-1}^i \cup \{e_j^i\}$ . This yields sets  $S_j^i$  with

$$d_{\mathcal{F}_{i-1}}(S_j^i) \geq k^{2k/3-j}. \quad (8)$$

Having specified the first  $\lfloor k/3 \rfloor$  vertices in  $e_{i+1}$ , we will now work to specify the remaining vertices. To this end, let  $D_{\mathcal{F}_{i-1}}(S_{\lfloor k/3 \rfloor}^i) := \{f \in \mathcal{F}_{i-1} : f \supseteq S_{\lfloor k/3 \rfloor}^i\}$ . Define

$$P_i := \{x \in X \setminus S_{\lfloor k/3 \rfloor}^i : x \in f \text{ for all } f \in D_{\mathcal{F}_{i-1}}(S_{\lfloor k/3 \rfloor}^i)\}. \quad (9)$$

It follows from (8), (9), and Lemma 5 that

$$k^{2k/3 - \lfloor k/3 \rfloor} \leq d_{\mathcal{F}_{i-1}}(S_{\lfloor k/3 \rfloor}^i) = d_{\mathcal{F}_{i-1}}(S_{\lfloor k/3 \rfloor}^i \cup P_i) \leq k^{k - \lfloor k/3 \rfloor - |P_i|}. \quad (10)$$

The inequality in (10) establishes that  $|P_i| \leq k/3$ . We now take  $e_{i+1}$  to be any edge in  $D_{\mathcal{F}_{i-1}}(S_{\lfloor k/3 \rfloor}^i)$ ; such an edge is guaranteed to exist since the set is non-empty by (10). Label

$$e_i = \{e_1^i, e_2^i, \dots, e_k^i\}$$

so that  $\{e_1^i, e_2^i, \dots, e_{\lfloor k/3 \rfloor}^i\} = S_{\lfloor k/3 \rfloor}^i$  and  $P_i \subseteq \{e_{\lfloor k/3 \rfloor + 1}^i, e_{\lfloor k/3 \rfloor + 2}^i, \dots, e_{2k/3}^i\}$ .

The question  $\Omega_i$  asks for the least integer  $\omega_i$  such that  $e_{\omega_i}^i \in e$ . Let  $V_i := V_{i-1} \cup \{e_{\omega_i}^i\}$ . We now consider two cases depending upon the answer  $\omega_i$ .

If  $\omega_i > 2k/3$ , then Guesser must ensure that the answer  $\omega_{i+1}$  to the next question will satisfy  $\omega_{i+1} \geq k/3$ . Observe that since  $\omega_i > 2k/3$ , it follows that  $S_{\lfloor k/3 \rfloor}^i \cap e = \emptyset$ . Also, since  $\omega_i > 2k/3$ , we have  $e_{\omega_i}^i \notin P_i$ . Hence, by the definition of  $P_i$  (see (9)), there exists an edge  $e_{i+1} \in D_{\mathcal{F}_{i-1}}(S_{\lfloor k/3 \rfloor}^i)$  with  $e_{i+1} \not\supseteq e_{\omega_i}^i$ . Label the vertices of this edge

$$e_{i+1} = \{e_1^{i+1}, e_2^{i+1}, \dots, e_k^{i+1}\}$$

so that  $\{e_1^{i+1}, e_2^{i+1}, \dots, e_{\lfloor k/3 \rfloor}^{i+1}\} = S_{\lfloor k/3 \rfloor}^i$ . It follows that  $e_{i+1} \cap V_i = \emptyset$ . The answer to question  $\Omega_{i+1}$  (based upon the testing edge  $e_{i+1}$ ) will identify a new vertex in  $e$  and necessitate an answer  $\omega_{i+1} \geq k/3$ .

In the complementary case  $\omega_{i+1} \leq 2k/3$ , the question  $\Omega_{i+1}$  must identify a new vertex in  $e$  and the answer  $\omega_{i+1}$  can be any integer in  $[k]$ . To accomplish this, Guesser takes the testing edge  $e_{i+1}$  to be any edge that does not intersect  $V_i$ ; such an edge exists since  $\tau(\mathcal{F}) = k$ .  $\square$

By Claim 6, Guesser may ask questions,  $\Omega_1, \Omega_2, \dots, \Omega_t$  that necessitate a sequence of answers  $\omega_1, \omega_2, \dots, \omega_t$  satisfying property (7). Moreover, after asking these questions, Guesser will have determined  $V_t \subset e$  with  $|V_t| = t$ . For the remaining  $k - t$  questions, Guesser will only require that each answer is a number in  $[k]$  that identifies a new vertex in  $e$ . This is possible by Claim 7.

Hence, after  $k$  questions are asked, Guesser will have determined the edge  $e$  selected by Chooser. It follows that the size of  $|\mathcal{F}|$  is bounded above by the number of sequence



$\omega_1, \omega_2, \dots, \omega_k \in [k]^k$  that satisfy property (7). Because the number of ways to select a pair  $\omega_i, \omega_{i+1} \in [k]$  with the condition in (7) is less than

$$k^2 - (k/3 - 2)(k/3 - 2) = (8/9)k^2 + 4k/3 - 4 < e^{-1/10}k^2$$

for  $k$  sufficiently large, it follows that

$$|\mathcal{F}| \leq \left(e^{-1/10}k^2\right)^{t/2} k^{k-t} = e^{-t/20}k^k \leq ek^{k-\alpha}$$

for  $k$  sufficiently large. This completes the proof of Lemma 2.

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## REFERENCES

- [1] P. Frankl, K. Ota, and N. Tokushige, *Covers in uniform intersecting families and a counterexample to a conjecture of Lovász*, J. Combin. Theory Ser. A **74** (1996), no. 1, 33–42. [↑1](#)
- [2] P. Erdős and L. Lovász, *Problems and results on 3-chromatic hypergraphs and some related questions*, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II, 1975, pp. 609–627. Colloq. Math. Soc. János Bolyai, Vol. 10. [↑1](#), [2](#)
- [3] Z. Tuza, *Inequalities for minimal covering sets in set systems of given rank*, Discrete Appl. Math. **51** (1994), no. 1-2, 187–195. 2nd Twente Workshop on Graphs and Combinatorial Optimization (Enschede, 1991). [↑1](#)

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